Comment on "Adaptive steady-state stabilization for nonlinear dynamical systems"

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(Received 29 March 2009; revised manuscript received 23 October 2009; published 18 March 2010)

Instead of the problematic reasoning presented in the work of D. J. Braun [Phys. Rev. E **78**, 016213 (2008)], a rigorous argument is provided to show the validity of the adaptive controller proposed by Braun under some particular assumptions. Without these assumptions, this controller may be failed to stabilize the unsteady state, which is numerically shown by specific examples. Also, the choice of parameters to guarantee the validity of these assumptions is illustrated.

DOI: 10.1103/PhysRevE.81.038201 PACS number(s): 05.45.Gg

Stabilization of unstable equilibrium in nonlinear or even chaotic dynamical systems via the adaptive controlling technique has become a focal topic of great importance in the field of chaos control and complex network synchronization. Among all the works on this topic, the recent work by Braun [1] proposed an interesting controller, which takes a full advantage of the adaptive technique. As was claimed, the controller can stabilize the unstable equilibrium without requiring any explicit knowledge of its position and even the original system dynamics. However, there is a flaw in the analytical verification of this proposed controller. This flaw manifests that the LaSalle invariance principle [2] cannot be directly used in the proof, and that Braun's controller may be failed to stabilize some concrete systems.

In [1], the author used the following argument: "since $f(x)$ is locally Lipschitz, it is bounded on its domain D, which implies $\exists l<\infty$ such that $\forall i$, $|f_i(x)|<l|x_i-y_i|$ for $\forall x\in D$ if $x_i\neq y_i$." Clearly, this argument is wrong even when $f_i(x)$ is a linear function and x belongs to any bounded set. For example, considering any bounded set $D\in\mathbb{R}$ and $f(x)=\theta x$ with $\theta\neq 0$, one cannot find any finite constant l such that $|f(x)|/|x-y|=|\frac{\theta x}{x-y}|<l$ for any $x\neq y\in D$, because the two-variable function $g(x,y)=|\frac{x}{x-y}|=|\frac{1}{1-y/x}|$ is an infinitely large quantity in the bounded D even if $x\neq y$. Thus, l, if existing, is $+\infty$. This implies that V with any finite L, as constructed in [1], is not a Lyapunov function, i.e., $\dot{V}\leq 0$ is not valid, and that the LaSalle invariance principle cannot be adopted yet. Therefore, the problematic argument in [1] leads to the failure of the proof as well as to the failure of the stabilization in some specific numerical simulations. However, in light of dynamical systems theory, we provide proof for the validity of this controller under some assumptions as follows.

Consider those bounded trajectories generated by the adaptively controlled systems proposed by Braun

$$\dot{x}_i=f_i(x)-k_i(x_i-y_i),$$

$$\dot{y}_i=\lambda_i(x_i-y_i),$$

$$\dot{k}_i=\alpha_i(x_i-y_i)^2,\quad i=1,2,\ldots,n, \tag{1}$$

with the continuous vector field $f_i(x)$ and the variable $s=[s_1,\ldots,s_n]^\top\in\mathbb{R}^n$ in which $s=x$, y, or k. Here, λ_i and α_i are parameters as described in [1]. For each bounded trajectory $\gamma(t;x^0,y^0,k^0)=[x(t;x^0,y^0,k^0),y(t;x^0,y^0,k^0),k(t;x^0,y^0,k^0)]$ with the initial data $[x^0,y^0,k^0]$, we have the existence of $\lim_{t\to\infty}k(t)=k^*$ because of the monotonicity and boundedness of the coupling gain variable $k(t)$. Then integrating the derivative of this variable yields:

$$\int_0^{+\infty}[x_i(s)-y_i(s)]^2\mathrm{d}s<+\infty.$$

This implies that each $x_i(t)-y_i(t)$ is L^2-integrable. Furthermore, we have the equation:

$$\dot{x}_i-\dot{y}_i=f_i(x)-(k_i+\lambda_i)(x_i-y_i).$$

Then, $\dot{x}_i-\dot{y}_i$ is uniformly bounded due to the continuity of the vector field and the boundedness of the trajectory as mentioned above. This, together with the L^2 integrability and continuity of $x_i(t)-y_i(t)$, yields that the error $x_i(t)-y_i(t)\to 0$ as $t\to+\infty$. Moreover, since the bounded trajectory is generated by the autonomous system (1), the Omega limit set

$$\Omega([x^0,y^0,k^0])=\{[x^*,y^*,k^*]\in\mathbb{R}^{3n}|\gamma(t_n;x^0,y^0,k^0)\to[x^*,y^*,k^*],\{t_n\}_{n=0}^{\infty}\to+\infty,n\to\infty\}$$

of the bounded trajectory is nonempty, connected, and invariant [3]. In particular, restricted in the Omega limit set, $s_i(t;x^*,y^*,k^*)$ is a constant function for each $s=x,y,k$ due to $\dot{s}_i(t)\equiv 0$ in the invariant limit set. Thus, $\Omega([x^0,y^0,k^0])=\{[x^*,y^*,k^*]\in\mathbb{R}^{3n}|f_i(x^*)=0,x^*=y^*\}$. Since the limit set is connected, $\Omega([x^0,y^0,k^0])$ only contains single equilibrium if *those equilibria of uncontrolled nonlinear system separately lie in* $\mathbb{R}^n$ (ESL). Therefore, under the assumption (ESL), both $x(t)$ and $y(t)$ tend toward the equilibrium, which means the controller is feasible for all *bounded* trajectories of system (1).

If the boundedness of all the trajectories of the controlled system (1) cannot be guaranteed, the stabilization with Braun's controller may be very slow and even failed though the dynamical behavior of the uncontrolled system is

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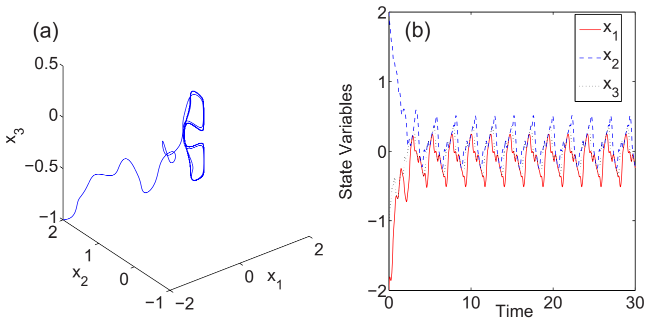

FIG. 1. (Color online) The dynamical behavior of the uncontrolled system (2) in the x_1-x_2-x_3 phase plane (a) and in the state variables versus time plane (b) with the initial value $[-2,2,-1]^{\top}$. Here and throughout, the MATLAB tool ODE45 is utilized to solve the continuous system.

bounded and the corresponding vector field is globally Lipschitzian. To illustrate this, we consider the following three-dimensional ordinary differential equations as an uncontrolled system:

$$\dot{x}_i = -x_i + \sum_{j=1}^{3} w_{ij} \sin(\alpha x_j), \quad i = 1,2,3, \tag{2}$$

where the symmetric connection matrix is select to be

$$W = \{w_{ij}\}_{3\times 3} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix},$$

and the parameter $\alpha=10$. System (2) can be regarded as a particular kind of artificial neural network model with sinusoidal activation functions. With the above parameters, system (2) is globally Lipschitzian and exhibits periodic oscillation, as shown in Fig. 1. According to system (2) and Braun's controller, we design the adaptively controlled system by

$$\dot{x}_i = -x_i + \sum_{j=1}^{3} w_{ij} \sin(\alpha x_j) - k_i(x_i - y_i),$$

$$\dot{y}_i = \lambda_i(x_i - y_i), \;\; \dot{k}_i = \alpha_i(x_i - y_i)^2, \quad i = 1,2,3, \tag{3}$$

where the parameters are taken as $\lambda_i=2$ and $\alpha_i=5$ for each i in the following numerical simulations. Contrary to the analytical result given in [1], the stabilization is unsuccessful with particularly selected parameters and initial values. As shown in Fig. 2, the controlled state variables x_i and the estimators y_i, though approaching synchronization, are not be stabilized to any equilibria of system (2) through the duration of simulation. In particular, as depicted in Fig. 2, those coupling gain variables k_i, though increasing slowly and monotonically, are not surely controlled by some upper bounds. Moreover, as shown in Fig. 3, each quantity

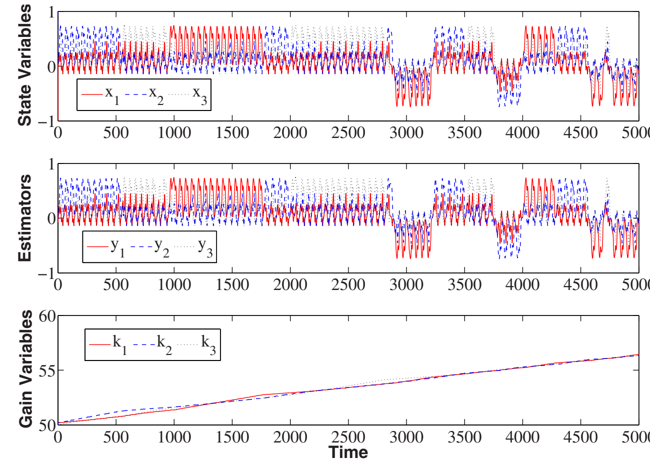

FIG. 2. (Color online) The dynamical behavior of the controlled system (3) shows an unsuccessful stabilization of system (2) through the duration of simulation. Here, the initial values for controlled system (3) are taken as $[-2,2,1,0,0,0,50,50,50]^{\top}$.

$$l_i = \left[-x_i + \sum_{j=1}^{3} w_{ij} \sin(\alpha x_j) \right] \Bigg/ (x_i - y_i),$$

which is defined as the Lipschitz constant in [1], can be tremendously huge. Therefore, the boundedness of all the trajectories of system (1) including the coupling gain variables k_i is crucial to a successful stabilization with Braun's controller. Without guaranteeing this boundedness, the stabilization probably can be very slow and even failed in practice.

It is noted that the boundedness requirement is also necessary in any stability analysis involving the use of the LaSalle invariance principle. Without verifying or assuming, *a priori*, the boundedness of the whole system, one cannot use the LaSalle invariance principle. In [1], the only assumption on the boundedness of the uncontrolled and controlled state variables x is inadequate, because the whole system also includes the estimators y and the coupling gain variables k.

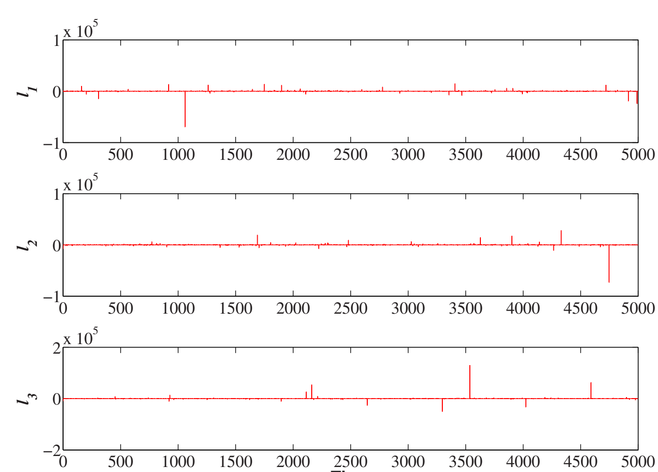

FIG. 3. (Color online) The variation in each l_i with the time evolution along the trajectories shown in Fig. 2.

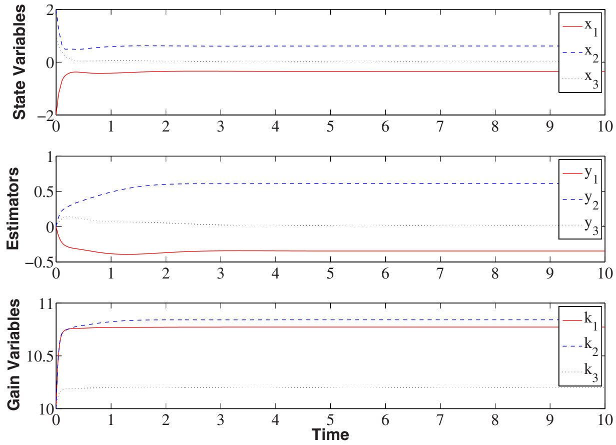

FIG. 4. (Color online) Successful stabilization of the controlled system (3) with initial values $k_i(0)=10$. The initial values for the state variables x_i and estimators y_i are the same as those in Fig. 2.

Besides, the boundedness assumption on the whole system and the LaSalle invariance principle together cannot ensure the validity of the Braun's controller, since we have explained that V is not a Lyapunov function. Therefore, as performed above, more elaborate analysis in light of dynamical systems theory should be adopted to verify this validity.

In general, the direct proof of the boundedness of some of or all of the trajectories produced by the controlled system (3) (not by the uncontrolled system) is really hard; however, it can be verified for concrete system either numerically or theoretically. To ensure the existence of the bounded trajectories and the stability of the controlled goal in practice, the initial value of k_i and the values of λ_i and α_i should be particularly adjusted. For the controlled system (2), we have shown unsuccessful stabilization above; however, we can also choose $k_i(0)=10$ to numerically guarantee the boundedness, and then get a practical stabilization (see Fig. 4). Furthermore, to clearly illustrate the role of λ_i, we simply take the linear function $f(x)=\theta x$ as an example. Then, $x^*=0$ is the equilibrium of the uncontrolled system, and the corresponding Jacobian matrix of the controlled system around the equilibrium $[0,0,k^*]$ is

$$\mathcal{J}=\begin{bmatrix} \theta-k^* & k^* & 0 \\ \lambda & -\lambda & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

in which k^* is a positive number. Clearly, aside from the zero eigenvalue, the other two eigenvalues of $\mathcal{J}$ being strictly negative becomes the condition for the stability of the equilibrium $[0,0,k^*]$. In fact, once this condition is satisfied, both $x(t)$ and the error $x(t)-y(t)$ are bounded and converging exponentially in the vicinity of the equilibrium, so that $k(t)=k(0)+\int_0^{+\infty}[x(s)-y(s)]^2\mathrm{d}s$ is convergent to some k^*. To guarantee the validity of this condition, we must have $\theta-\lambda-k^*<0$ and $\theta\lambda<0$. Since k^* can be an arbitrarily positive number dependent on $k(0)$, the first inequality is always valid for any pair of θ and λ. If $\theta>0$ and thus $x^*=0$ is an unstable equilibrium of the uncontrolled system, λ must be taken as a negative number to satisfy the latter inequality; if $\theta<0$, λ must be positive. Mathematically, the above result based on the linearization analysis is only valid for the initial data in the vicinity of the equilibrium; however, numerical simulations manifest that the result still holds away from the equilibrium. It is noted that the effect of the sign of λ has been numerically exerted in the saddle stabilization of the Lorenz system in [1], where, however, its effect on the boundedness of the controlled systems was not analytically discussed.

Moreover, if the assumption (ESL) is violated, the uncontrolled system has continuous equilibria. Theoretically, the convergence of $x(t)$ and $y(t)$ cannot be guaranteed, and both variables may slowly fluctuates among some interval though each $x_i(t)\to y_i(t)$ as $t\to+\infty$. This means that the proposed controller may be infeasible to stabilize the unstable equilibrium in such a case. To be candid, in practice, the convergence of $x(t)$ and $y(t)$ is always valid numerically, and then both $x(t)$ and $y(t)$ surely converge to the points embedded in the continuous equilibria.

This Comment has pointed out a flaw existing in the analytical argument in [1]. It is the flaw that motivates us to produce more assumptions and conditions ensuring the validity of Braun's controller. Appropriately adjusting the parameters and initial values to meet those assumptions makes Braun's controller more useful in chaos control. Moreover, it is valuable to mention that numerical verification can only show boundedness of orbits and successful or failed stabilization in a finite duration. More practical conditions as well as more convincing counterexample with theoretical arguments are expected.

This research was supported by the NNSF of China (Grant Nos. 10501008 and 60874121) and by the Rising-Star Program Foundation of Shanghai, China (Grant No. 07QA14002).

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